LiABlock_3D: A software tool for collapse mechanism analysis of historic masonry structures

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ABSTRACT

A discrete rigid block model is proposed for collapse mechanism analysis of three-dimensional historic masonry structures subjected to point live loads, seismic induced lateral loads and settlements. The model is made of polyhedral rigid blocks interacting at no-tension, frictional contact interfaces and can be used to represent complex assemblages and bond patterns. The formulation and the solution procedure of the underlying limit equilibrium analysis problem were implemented in LiABlock_3D, a MATLAB based tool with Graphical User Interface (GUI). The software was designed to import the geometric model from commercial Computer Aided Design tools, thus allowing high flexibility of structural configurations and masonry patterns. The graphical interface is also used to define material properties as well as boundary and loading conditions. Numerical and experimental case studies from the literature were analysed to show the ability of the model developed in predicting the collapse behaviour of a variety of structural typologies. Those include arches, vaults and domes under vertical and horizontal live loads and spreading supports. A two-storey masonry building with a barrel vault at first level is also analysed under variable lateral loads and support movement. Potentialities and limitations of the proposed formulation and tool are discussed on the basis of the results obtained and also in terms of computational efficiency.

Keywords: Historic masonry structures; Collapse mechanism analysis; Live loads; Moving supports; Discrete rigid block model; Polyhedral rigid blocks; Mathematical programming; Graphical User Interface.

1 INTRODUCTION

Historic masonry structures subject to earthquakes or settlements frequently exhibit local collapse mechanisms which involve in-plane and out-of-plane response of structural elements [1, 2].

The analysis of local failure mechanisms is a topic which is still debated in the literature and involves several issues mainly related to the discontinuous nature of masonry and the interaction between structural parts [3, 4]. To address these issues several modelling approaches have been developed to date [5-7].

Discrete modelling approaches based on finite, distinct or rigid body spring models such as those presented in [8-19] can be conveniently used to analyse the activation and the evolution of failure mechanisms in historic masonry structures with non-linear static or dynamic analysis.

As an alternative to the above mentioned approaches, simple assessment methods based on limit analysis of collapse mechanisms can also be used for estimating the vulnerability of local failure modes [20, 21].

The collapse mechanism limit analysis of masonry structures is based on the discretization of the structure into an assemblage of rigid bodies, usually interacting at no-tension, frictional contact interfaces. The evaluation of the load factor corresponding to the activation of the rigid block mechanism is carried out through the application of the limit analysis theorems.

Several analytical formulations are available in the literature for the calculation of load factors that activate local failure mechanism of masonry structures, which mainly depend on the limit analysis formulation adopted (lower and upper bound theorems) and on the assumptions on frictional behaviour when also sliding failure is taken into account [22]. When using analytical formulations, structural systems are usually idealized as assemblages of rigid macro-elements which represent entire portions of masonry walls separated by the potential crack lines associated to the collapse mechanism. Giuffré was among the first to propose the use of the macro-element approach for the assessment of the seismic capacity of masonry buildings [23]. More recently Lagomarsino [24] extended the macro-

element approach to the seismic verification of monumental buildings subjected to rocking. The verification is carried out by comparing the spectral accelerations that activate the mechanism with the seismic demand according to a linear or non-linear kinematic approach.

Various tools have been developed in the recent years from these analytical formulations with the aim to support the technical community with simple and efficient softwares for the seismic verification of existing structures [8]. Among those, the application *CINE*, developed by Milano et al. [25] is a collection of spreadsheets for the evaluation of load factors activating 11 out of plane local mechanisms of masonry buildings based on linear kinematic analysis.

In this line, another useful tool is the software *c-Sisma*, developed by Modena et al. [26] within the framework of the Italian ReLUIS project [27]. *c-Sisma* includes 20 kinematic models for the calculation of load factors corresponding to pre-determined out-of-plane mechanisms of masonry walls. The *c-Sisma* interface allows the user to define specific input data in terms of geometry of the walls, material parameters, seismic loads and their combination and provides as output the inertial mass multiplier which activate the mechanism and carries out the seismic verification according to the national building code [28-29].

Examples of interactive and parametric analytical tools based on thrust line method for limit analysis of vaulted masonry buildings subjected to live loads and imposed displacement at the supports can be found in [30-32].

Although the above mentioned formulations and tools can be conveniently used for the assessment of local failure modes in masonry structures, the need to identify a-priori the collapse mechanisms and their idealization into few, predetermined rigid macro-blocks might limit the range of applications and the structural types which can be analysed with these softwares.

As an alternative to software tools based on analytical formulations, several studies addressed the possibility to use finite element and computational rigid block models for limit analysis of collapse mechanism in masonry structures [33-39]. Computational limit analysis has been receiving a renewed

interest in the last years thanks to the development of robust and fast optimization algorithms for the solution of the large-scale mathematical programming problems which stem from limit analysis formulations [40].

A computational tool which is based on a finite element discretization constituted by NURBS rigid elements joined by elasto-plastic interfaces was recently presented in Chiozzi et al. [41].

Software tools based on rigid block modelling for limit analysis are also available. Among those, the *LimitState:RING* software [42] is devoted to the analysis of 2D masonry bridges under vertical live loads and settlement. The software is based on the formulation developed by Gilbert et al. [36] and provides as outputs the critical failure modes and the thrust zone at collapse. Examples of applications to masonry vaults can be found in [43].

As for 3D masonry assemblages, it is worth mentioning the software tool *Block* developed by Orduña [44] on the basis of the formulation presented in [45] and on the solution of the mixed complementarity mathematical problem (MCP) which stem from the coupled system of static and kinematic equations governing the behaviour of the rigid block assemblage [46].

The formulations underlying previous software tools for rigid block limit analysis (RBLA) use a surface (convex) model to idealize contact interactions at no-tension frictional interfaces. Although the use of surface modelling for contacts in RBLA has been recognized as an accurate approach for the collapse mechanism analysis of masonry structures, drawbacks in its extensive application raised so far due to the computational efforts and modelling challenges required to consider large 3D assemblages [47].

A simple alternative formulation has been recently adopted in [48] for 3D assemblages made of rectangular rigid blocks. The formulation is based on the point contact model which noticeably simplify the definition of failure criteria at interfaces. The model makes use of conic programming for the solution of the underlying limit analysis problem, which reduces computational time dramatically, and is also able to take into account a non-dilatant (non-associative) sliding behaviour. The

formulation firstly presented in [48] was after that refined in order to take into account limited compressive strength [49] and was also applied to the analysis of collapse mechanisms induced by support movements [50].

Starting from the previous works, a novel formulation using polyhedral rigid blocks is presented in this paper for collapse mechanism analysis of 3D assemblages subjected to variable loads and support movements. The proposed formulation was implemented in a custom-made software tool, LiABlock_3D, which is characterized by a simple and interactive graphical user interface for the definition of the input data, the selection of the analysis type (i.e. live loads or moving supports) and visualization of results. As such, the developed tool extends the formulations presented in [48, 50] to the general case of 3D assemblages made of both regular and irregular discrete rigid blocks, and combines into a single package the possibility to perform different analysis types. The proposed software provides an enhanced flexibility in terms of potential field of application and relies at the same time on the computational efficiency and accuracy of the original solution procedures.

The paper is organized as follows. Section 2 introduces the theoretical background and the architecture of LiABlock_3D, highlighting the peculiarities of the novel rigid block model, the mathematical formulation of the limit analysis problem, and how these concepts have been implemented in the graphical user interface. Section 3 presents validation examples on numerical case studies from the literature dealing with arches and vaults subjected to variable vertical point loads and spreading supports. In Section 4, two experimental case studies comprising a hemispherical masonry dome and a block cross vault are analysed. Then, in Section 5 a two-storey masonry building under variable later loads and foundation settlement is considered to illustrate the computational efficiency of the developed software.

2 SOFTWARE ARCHITECTURE AND RIGID BLOCK MODEL

LiABlock_3D is a novel software tool for the limit equilibrium analysis of masonry structures modelled as assemblages of three-dimensional rigid blocks subjected to live loads and settlements [51].

The main fields of application are monumental buildings made of large masonry blocks or stones and masonry assemblages with regular or complex bond patterns, such as arched structures and vaults, where the contribution of mortar can be neglected (Fig. 1).

LiABlock_3D is deployed as a standalone executable application from MATLAB scripts and functions originated by the first and third author.

FIGURE 1. Application examples of LiABlock_3D to historic masonry structures under (a) lateral loads and (b) settlement.

The LiABlock_3D software architecture is composed by three sequential modules, the *model builder*, the *solver*, and the *output module*.

A simple graphical user interface (GUI) was created in the MATLAB GUI development environment in order to collect the three modules in a single window interface, which can be also used to import the assemblage configurations from three-dimensional CAD models (Fig. 2).

The interactive command window guides the user to the definition of the numerical model of the masonry assemblage, the selection of input data for the analysis, the calculation of the solution and the visualization of results. Editable values are reported either in the white boxes, where the user can directly enter the numerical values, or in the drop-down list of pop up menus, where the user makes a choice among a list of mutually exclusive values.

FIGURE 2. LiABlock_3D: a) Software modules; b) The command window interface.

A description of the software modules and of the underlying rigid block model formulation is provided in the following sections.

2.1 Model builder

2.1.1 The model geometry

Liablock_3D is able to analyse 3D structures made of polyhedral blocks in contact by means of quadrilateral interfaces. The routine implemented consists in creating the geometry of the assemblage in AutoCAD, exporting relevant data to a Microsoft Excel spreadsheet and importing them in LiABlock_3D for the generation of the numerical model.

For a given 3D block structure, the first steps are to identify the block types used to discretize the structural assemblage (i.e. rectangular and polyhedral blocks) and the associated contact interfaces. Block types are modelled as Autodesk CAD blocks with attributes. The information required for each block type are the Cartesian coordinates of the vertexes of the polyhedron, which define the geometry, its centroid and the coordinates of vertexes associated to each contact interface. In the CAD block editor, object ‘points’ are used to represent the above-mentioned points and position attributes are assigned to them. Additional attributes are attached to the block types to store the block volume and the labels of contact interface.

Figure 3 reports an example of block types identification for a simple portal arch made of regular polyhedral blocks. In such a case, the definition of three block types only is required to model the entire structure. In particular, a voussoir block (type A in Fig. 3b) with eight primary vertexes and two contact interfaces is created to model the arch. Two rectangular block types are created to model the arch springers (type B in Fig. 3b with eight primary vertexes, ten potential contact points and three interfaces) and piers (type C in Fig. 3b with eight primary vertexes and potential contact points and two interfaces) respectively. To model blocks with more than four sides per each base, an additional attribute can be defined, named ‘base’, where a different number of vertexes can be assigned.

Once the basic blocks with attributes are defined, the model geometry is created by simply assembling the block types in the AutoCAD model space. By launching the data Extraction wizard of AutoCAD,

the creation of an output file starts. The result is a spreadsheet with as many rows as many polyhedral blocks are in the model and as many columns as the attributes defined (see Table 1).

FIGURE 3. CAD model of a simple portal arch made of polyhedral blocks (a) and block types with contact interfaces and attributes (b).

Table 1. The scheme of the Excel spreadsheet obtained using the AutoCAD data Extraction wizard for each block type of the portal arch represented in Figure 3.

Block type	POINT_1	…..	POINT_8	POINT_9	POINT_10	CONTACT_1	CONTACT_2	CONTACT_3	CENTROID	VOL.
Type_A	x_{A1}, y_{A1}, z_{A1}	…	x_{A8}, y_{A8}, z_{A8}	*empty cell*	*empty cell*	1, 2, 3, 4	5, 6, 7, 8	*empty cell*	x_A, y_A, z_A	V_A
Type_B	x_{B1}, y_{B1}, z_{B1}	…	x_{B8}, y_{B8}, z_{B8}	x_{B9}, y_{B9}, z_{B9}	x_{B10}, y_{B10}, z_{B10}	1, 2, 3, 4	6, 7, 9, 10	9, 10, 5, 8	x_B, y_B, z_B	V_B
Type_C	x_{C1}, y_{C1}, z_{C1}	…	x_{C8}, y_{C8}, z_{C8}	*empty cell*	*empty cell*	1, 2, 3, 4	5, 6, 7, 8	*empty cell*	x_C, y_C, z_C	V_C

The first row reports the labels of attributes (extracted in any order). From the second row on, all the information created in the form of attributes will be reported for each block composing the assemblage, namely: block type, coordinates of points used to define the geometry of the polyhedral blocks and associated to contact interfaces, coordinates of the centroid and block volume.

The creation of the geometry of the model is completed by importing in LiAblock_3D the data reported in the spreadsheet. By pushing the '*Browse file*' button, a standard routine acquires the data from the spreadsheet and allocates them in an internal structure array with predefined fields. By clicking the '*Plot geometry*' button, an interactive Matlab figure opens and reports the 3D plot of the

model geometry. Block numbers can be displayed by selecting the proper option in the popup menu label.

2.1.2 Generation of the numerical model

Once the geometry of the model has been created and imported in LiABlock_3D, the '*Generate model*' button is used to launch the routine that builds the equilibrium matrix $\boldsymbol{A}$ governing the limit analysis problem associated to the rigid block model.

FIGURE 4. a) Representation of a rigid block discrete model generated in LiABlock_3D; b) Static variables at contact point *k*; c) dead and live loads at block *i*.

The coefficients of the equilibrium matrix are determined assuming a 'point' contact model for interactions, with internal forces (i.e. contact or static variables) acting at each contact point *k* located at the vertices of the interface *j*. The internal forces are represented by the shear force components t_{1k}, t_{2k} along local coordinate axes and by the normal force n_k.

With static variables collected in the vector $\boldsymbol{x_k} = [t_{1k} \;\; t_{2k} \;\; n_k]^{\mathrm{T}}$, the contact equilibrium matrix associated to the 6 degrees of freedom (DOFs) of block *i* and to contact point *k* can be expressed as:

$$\boldsymbol{A_{ik}} = -\begin{bmatrix} \hat{\boldsymbol{t}}_{\boldsymbol{1k}} & \hat{\boldsymbol{t}}_{\boldsymbol{2k}} & \hat{\boldsymbol{n}}_{\boldsymbol{k}} \\ \boldsymbol{R_{ik}} \times \hat{\boldsymbol{t}}_{\boldsymbol{1k}} & \boldsymbol{R_{ik}} \times \hat{\boldsymbol{t}}_{\boldsymbol{2k}} & \boldsymbol{R_{ik}} \times \hat{\boldsymbol{n}}_{\boldsymbol{k}} \end{bmatrix}, \tag{1}$$

where: $\boldsymbol{A_{ik}}$ is a (6×3) matrix; $\hat{\boldsymbol{t}}_{\boldsymbol{1k}}$, $\hat{\boldsymbol{t}}_{\boldsymbol{2k}}$ and $\hat{\boldsymbol{n}}_{\boldsymbol{k}}$ are, respectively, the column vectors with components of tangent and normal to contact interface; $\boldsymbol{R_{ik}}$ is the position vector of contact point *k* with respect to the block centroid (Fig. 4b).

For the assemblage of matrix $\boldsymbol{A}$ corresponding to the whole structure, LiABlock_3D assumes that the discrete rigid block models are simply supported to the ground (support layer at z=0). Alternatively, support layers at different heights above ground level can be also imposed (i.e. in the case of flying arches). Once the support condition has been specified in the relevant box, the contacts among the blocks are automatically detected when the '*Generate model*' button is clicked, then sequentially numbered and a contact matrix, collecting the information about the contacts, is created.

2.1.3 Contact failure conditions

Two contact failure modes are implemented in LiABlock_3D: opening (separation) and sliding of interfaces at contact points *k*.

Under the assumptions of: *i)* infinite compressive strength; *ii*) tensionless behaviour; *iii)* shear failure governed by a Coulomb type criterion with isotropic friction and cohesionless behaviour, the contact behaviour is governed by two failure conditions, posed in the form:

$$-n_k \leq 0 \tag{2}$$

$$\sqrt{t_{1k}^2 + t_{2k}^2} - \mu\, n_k \leq 0 \tag{3}$$

where μ is the friction coefficient.

Previous failure conditions can be also posed in the form:

$$C_k = \left\{ \boldsymbol{x_k} \in \mathrm{R}^3 : \mu\, n_k \geq \sqrt{t_{1k}^2 + t_{2k}^2}\,,\ n_k \geq 0 \right\} \tag{4}$$

where C_k is a convex cone.

The contact failure conditions are automatically implemented in LiABlock_3D. The required input parameter is the friction coefficient μ.

2.1.4 Loading conditions

LiABlock_3D assumes that each rigid block *i* is loaded by dead loads $\boldsymbol{f_{Di}}$ and live loads $\boldsymbol{f_{Li}}$ which are applied to the centroid of each rigid block.

The sum of the dead loads and of the live load increased by an unknown scalar multiplier α represents the external loads $\boldsymbol{f_i}$:

$$\boldsymbol{f_i} = \boldsymbol{f_{Di}} + \alpha \boldsymbol{f_{Li}}. \tag{5}$$

where $\boldsymbol{f_i}$ is a (6×1) vector, with components associated to the 6 DOFs of block *i*.

By default, LiABlock_3D computes the self-weight of each rigid block on the basis of the unit weight of material ρ and block volume attribute V_i and assumes it as dead load $\boldsymbol{f_{Di}} = \boldsymbol{f_{Di}^{\rho}}$, where:

$$\boldsymbol{f_{Di}^{\rho}} = [0 \quad 0 \quad -\rho V_i \quad 0 \quad 0 \quad 0]^{\mathrm{T}}. \tag{6}$$

Additional dead loads of known magnitude can be applied to a selection of rigid blocks to simulate the effect of other permanent actions. Those include downwards vertical loads $\boldsymbol{f_{Di}^{fl}}$ to model the actions induced by floors and constant loads $\boldsymbol{f_{Di}^{c}}$, which can be applied along any global axis, to model the effect of other imposed forces, such as those associated to tie elements or static thrusts due to arches or vaults.

As such, the vector of dead loads is expressed as:

$$\boldsymbol{f_{Di}} = \boldsymbol{f_{Di}^{\rho}} + \boldsymbol{f_{Di}^{fl}} + \boldsymbol{f_{Di}^{c}}. \tag{7}$$

Live loads $\boldsymbol{f_{Li}}$ are expressed as a factor of block unit weight f_{Di}^{ρ} and floor loads f_{Di}^{fl}, and can be applied in any direction along the global coordinate axes, to the whole assemblage or to a selection of rigid blocks.

The available options for the floor loads, live loads and constant loads can be specified in the relevant editable white boxes and interactive pop-up menus.

The *"applied to"* boxes are used to define the rigid blocks to be loaded. Three loading options are available: to the whole assemblage typing '*all*'; to none of the rigid blocks typing '*none*'; to a selection of them, typing the blocks number.

As for the constant loads, a dedicated window opens when the relevant option is selected in the pop-up menu. Up to two different constant loads can be applied to simulate the prestress effects of ties, and their loading axis, direction, and value can be defined in the corresponding boxes.

LiABlock_3D includes also a module to evaluate the behaviour of masonry assemblages subjected to uniform settlements. In such a case, a support block *s* is used to introduce ground movement effects on the structure. The collapse load multiplier is used to vary the reaction at the support block (which is expressed as a function of dead loads) up to the activation of the failure mechanism [50].

In particular, the following expression is adopted for the external force $\boldsymbol{f_s}$ at the support block:

$$\boldsymbol{f_s} = (1-\alpha)\boldsymbol{f_{Ds}}, \tag{8}$$

being $\boldsymbol{f_{Ds}}$ a starting value of the base reaction which is automatically assigned in the tool as a function of the total dead load acting on the model.

FIGURE 5. Example of settlement analysis on a portal arch: a) geometrical model; b) failure mode.

To activate the settlement module in LiABlock_3D, the proper option in the relevant pop up menu has to be selected. In this case, it is assumed by default that the support block *s* corresponds to the last block number in the list of the Excel spreadsheet obtained using the AutoCAD data Extraction wizard from the geometric model of the masonry assemblage.

2.2 Model solver

LiABLock_3D provides a two-step procedure for the solution of the limit analysis problem.

The first step is the solution of the mathematical programming problem which arises from the formulation of the limit analysis problem when an associative friction behaviour is assumed. The assumption of associative behaviour involves dilatancy (i.e. normal displacement rates) when sliding occurs at a contact point (Figs. 6a, 6c). Under these conditions, the load factor is unique, and it can be calculated according to a static or kinematic formulation of the limit analysis problem. LiABlock_3D implements the lower bound problem of limit analysis according to the following formulation:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{f_D} + \alpha \boldsymbol{f_L} \\ & \boldsymbol{x} \in C \end{aligned} \tag{9}$$

where $\boldsymbol{x}$ is the vector of the internal static variables at contact interfaces, $\boldsymbol{A}$ is the equilibrium matrix, $\boldsymbol{f}$ is the vector of external loads and C is the convex cone.

In the above formulation the second and third expressions represent the equilibrium and failure conditions of the 3D rigid block assemblage, respectively. The Mosek optimization software [52] is used to solve the mathematical programming problem.

It is well known that the collapse load multiplier obtained under the assumption of associative friction represents an upper bound on the multiplier corresponding to non-associative friction model, i.e. to zero-dilatancy sliding behaviour (Fig. 6c). To take into account the non-associative frictional behaviour and to compute safe values of the collapse load multiplier, an iterative procedure is implemented which solves a series of cone programming sub-problems, according to [48].

Starting from the associative solution, values of normal forces computed at previous iterations are used to define a fictitious failure condition for which the associative (i.e. normal) flow rule leads to zero-dilatancy behaviour. By default, in order to reproduce a non-dilatant behaviour under the assumption of normality flow rule, LiABlock_3D assumes that sliding behaviour at contacts is governed at each iteration by a cylindrical failure surface with a zero angle of friction and an effective cohesion intercept calculated as follows (Fig. 6d):

$$c_{k,\mathrm{iter+1}} = \left(\beta n_{k,\mathrm{iter}} + (1-\beta) n_{k,\mathrm{iter-1}}\right)(1+\gamma)\tan\varphi_k + c_k^0, \tag{10}$$

where β is an algorithm parameter used to calculate normal forces $n_{k,\mathrm{iter+1}}$ at iteration no. iter+1 on the basis of normal forces computed at previous iterations, and c_k^0 is a small cohesion value which can be introduced to overcome numerical convergence problems, according to [48]. The parameter γ governs the angle of friction of the fictitious failure surface (posed equal to 0 to obtain a zero angle of friction in Fig. 6).

FIGURE 6. Modified failure surfaces used in the iterative solution: a) associative solution with dilatancy; b) non-associative solution with zero dilatancy; d) associative failure domain; e) non-associative failure domain.

To improve convergence characteristics in case of numerical problems, different values of the algorithm parameters β, γ and c_k^0 can be assigned in the window which is displayed when clicking on the button 'Set parameters'.

This windows also allows to modify the tolerance used to exit from the iterative solution procedure and the maximum number of iterations in the case of convergence problems.

2.3 Output and report

Utility functions are available to plot failure mechanism configurations increasing or decreasing the scale of displacement rates. Convergence plots, which report the load factor calculated at each iteration, are also available. Other functionalities involve the possibility to plot the geometric configuration of the structure by selecting '*Geo 'on"* or to highlight loaded block activating the colour menu.

Considering that limit analysis is based on the assumption of infinitesimal and indefinite displacements, it is clear that the plot of the failure mechanism is indicative of the collapse mode only rather than of the displaced configuration in terms of finite values of displacements [53, 54].

3 NUMERICAL EXAMPLES

In this section, numerical case studies of circular arches subjected to point variable loads and spreading supports are analysed with the aim to show the capability of the developed software in modelling curved masonry structures under different loading conditions.

The algorithm parameters β and γ for non-associative solution were taken as 0.6 and 0.0, the fictitious cohesion was equal to 1e-5×n_{max}, where n_{max} is the largest normal force calculated at a given iteration. The convergence tolerance was 10^{-3}, and a maximum number of iterations equal to 10 was used. The numerical analyses were carried out on a 3.50 GHz Intel Xeon Processor E5-1650 with 16.0 GB of RAM.

3.1 Circular arches and vault subjected to vertical live loads

The circular arches and the barrel vault shown in Figure 7 were analysed for validation. The same case studies were also analysed in [55] using different formulations for the mathematical programming problems underlying the limit analysis problem and also using different solvers.

The friction coefficient used for numerical analysis is equal to 0.75 and the unit weight is 25.0 kN/m^3.

FIGURE 7. a) Circular arch dimensions; b) Circular arch collapse mechanism; c) Two-ring arch dimensions; d) Two-ring arch collapse mechanism; e) Barrel vault dimensions; f) Barrel vault collapse mechanism.

The first case study is a circular arch subjected to a point live load (Fig. 7a). The arch is made of 16 voussoir blocks with 17 rectangular contact interfaces. The variable point load is assigned as a factor of the self weight of the loaded block.

The second case study is a double ring voussoir arch. The dimensions of the inner ring are the same of the single ring arch. Contact points are used along the surfaces between the two rings. Also in this case a vertical, variable point load is applied to the outer ring.

The third case study is a barrel vault with dimensions and block arrangement shown in Fig. 7e. Two block types were defined in AutoCAD to generate in LiABlock_3D the entire geometry of the vault, which comprises 63 full rigid blocks, 18 half blocks and 215 rectangular contact faces. The vault is loaded by variable forces distributed on four blocks as indicated in Fig. 7e.

The predicted failure mechanisms are shown in Fig. 7b, 7d, 7f and corresponding load factors are reported in Table 2.

In all cases the collapse occurs by a four-hinge failure mechanism. In the case of the single ring arch and the barrel vault, the values of the collapse load factors for the associative and non-associative solution are the same, due to the nature of the failure mode activated, which in this case does not involve sliding at contact points. A difference of 6% between the non-associative and associative solution is obtained in the case of the double ring arch due to the sliding at the contact interfaces between the two rings. The plot of the convergence behaviour for this case study is showed in Fig. 8, where the load factor computed at each step of the iterative solution procedure is reported, iteration no. 0 corresponding to the associative solution.

The comparison of results with those obtained in [55] shows a perfect agreement in the case of the single ring arch, both in terms of failure mechanism and load factor. In contrast, a remarkable difference of the results between LiABlock_3D and those reported in [55] can be noted for the other case studies. In the case of the barrel vault, this is because a different failure mechanism is obtained in [55], with different position of the hinges. However, in the interests of safety, it should be noted that the lower value of the collapse load multiplier obtained with LiABlock_3D was also validated against the analytical solution which was obtained using rigid body mechanism analysis and virtual work principle.

The differences observed in the case of the double ring arch can be ascribed to the different formulation of the limit analysis problem and solver adopted in [55].

Table 2. Masonry arches and vaults under vertical variable loads Comparison of numerical results.

Model	Model size (b × c)	α [55]	Associative		Non-associative formulation		Diff. α
			α_{assoc}	CPU Time (s)	$\alpha_{non\ assoc}$	CPU Time (s)	%
Circular arch	16 x 68	4.45	4.45	1.14	4.45	0.05	0.0
Two rings arch	33 x 268	17.29	15.12	0.03	14.19	0.18	17.9
Barrel vault	81 x 860	3.06	2.48	0.06	2.48	0.18	18.9

FIGURE 8. Two rings arch: collapse load factor vs. iteration.

The case study of a circular arch subject to eccentric load was also considered to point out the ability of the proposed formulation to capture 3D behaviour associated to torsion failure on curved assemblages. The arch configuration and dimensions were inspired to the case studies presented in [35-55]. The model is made of four blocks subject to dead loads applied to the centroid of each block and corresponding to unit weight (Fig. 9a). An additional rigid block was generated to apply an eccentric live load according to the configuration presented in [55] and expressed as a factor of the unit weight of the loading block.

The results of the numerical analysis obtained when varying the friction coefficient in the range 0.50-0.20 are shown in Fig. 9b. A four hinges failure mechanism is obtained when the friction coefficient is larger than 0.270. For lower values of the friction coefficient, twisting mechanisms occur which

involve two blocks or a single block as indicated in Fig 9b. The failure mechanisms obtained and collapse load multipliers are in a good agreement with those reported in [55].

FIGURE 9. Four block arch subjected to eccentric live load: a) Configuration; b) Sensitivity analysis to friction coefficient.

3.2 Circular arch subjected to spreading support

The case study of a circular arch on spreading supports was considered in order to validate the settlement module of the developed software.

The rigid block model developed for numerical simulation is shown in Fig. 10a. The mean radius of the arch is 220 mm and the radial thickness is 50 mm. The unit weight is 25.0 kN/m^3 and friction coefficient is equal to 0.7. The additional rigid block indicated in Fig. 10a is used to simulate the spreading support.

The dimensions and unit weight of the arch are the same of the voussoir arch tested in [30] under spreading supports till collapse. The same arch was also analysed in [56] to test the accuracy of other numerical and analytical formulations based on limit analysis theorems.

The predicted failure mechanism and the minimum thrust are reported in Fig. 10b and Table 3. Outward movement of the support block induces the opening of three hinges, which are associated to a stable rigid body mechanism. The predicted position of the three hinges and the value of the associated reaction $\boldsymbol{f_s}$ (i.e., the minimum thrust) are in full accordance with analytical results and outcomes of experimental tests presented in [30, 56].

FIGURE 10. a) Voussoir circular arch on spreading support; b) Predicted failure mechanism.

Table 3. Circular arch subjected to spreading support. Comparison of numerical and analytical results.

Model	Model size (b × c)	f_S Analytical (kN)	Associative f_S		Non-associative formulation		Diff. α
			$f_{S\ assoc}$ (kN)	CPU Time (s)	$f_{S\ non\ assoc}$ (kN)	CPU Time (s)	%
Circular arch	17 x 68	6.14	6.14	0.03	6.14	0.04	0.0

4 EXPERIMENTAL CASE STUDIES

Two experimental case studies were considered to validate the developed software against failure modes in masonry domes and vaults [57, 58]. The first case study is a masonry dome presented in [59-61]. The second case study is a block cross vault investigated by Rossi et al. under spreading supports [17, 62].

4.1 Masonry dome subjected to horizontal live loads

The dome considered is a small scale specimen made of dry-stacked plaster blocks [61]. The dimensions of the dome are shown in Fig. 11a. The exterior radius is equal to 164.0 mm and thickness is equal to 32.8 mm. The dome was tested on a tilting table in order to investigate failure mechanism and acceleration value promoting the collapse under a uniform distribution of lateral accelerations [61].

The rigid block numerical model is made of 137 blocks and 1276 contact points located along bed joint interfaces. The numerical model was generated in AutoCAD from 8 block types which were obtained by dividing the dome into seven rings and a central cap using polar angles as indicated in [61]. The value of friction coefficient was taken as 0.7, according to [60-62], and a unit weight of 12.5 kN/m^3 was used. To reproduce experimental loading conditions, each block was subjected to horizontal live loads expressed as a factor of the block unit weight.

The computed collapse load factor and predicted failure mechanism are presented in Table 4 and Figure 11b. The dome collapses with a five hinges failure mechanism along the meridian sections, also involving opening and sliding failures distributed along vertical and bed joint interfaces, respectively. The comparison with the experimental tests presented in [61] shows a quite good agreement with numerical results.

FIGURE 11. a) Hemispherical dome dimensions; b) Hemispherical dome collapse mechanism.

The angles of extrados and intrados hinge positions are slightly different from those observed experimentally [61]. For example, whereas the location of an intrados hinge was observed experimentally between the second and third row of blocks on the right end side of the dome, multiple hinge locations are predicted by the numerical model at the upper block rows. Moreover, significant

sliding failures were observed experimentally between lower block rows just before collapse, which can be also associated to the large displacement regime rather than to the onset (i.e. initiation) of the failure mechanism, where small displacement can be assumed in accordance to the numerical model.

The difference of numerical and experimental collapse load multipliers is 16.5 percent, which can be likely ascribed to the influence of geometric imperfections (i.e. corner rounding and construction) on test results.

In terms of computational efficiency, the numerical analyses took only a few seconds to obtain a solution, as reported in Table 4 for the associative and non-associative solution.

Table 4. Masonry dome under horizontal live loads: Comparison of numerical and experimental results.

Model	Model size (b × c)	α [61]	Associative		Non-associative formulation		Diff. α
			$\alpha_{assoc.}$	CPU Time (s)	$\alpha_{non\text{-}assoc.}$	CPU Time (s)	%
Dome	137×1276	0.460	0.589	0.9	0.536	2.4	16.5

4.2 Cross vault subjected to support movement

The experimental case study investigated in this section is the scale model of a cross vault tested under spreading supports presented in [62]. The specimen was made of plastic blocks with dry joints. The block unit weight was 27.0 kN/m^3 and the friction coefficient was 0.56. The specimen was tested under imposed displacements at the supports in order to reproduce a simple shear failure mechanism in the horizontal plane.

The numerical model of the cross vault is shown in Figure 12a and was generated on the basis of the CAD model used for 3D printing in [62]. It comprises 91 block types, 1131 blocks, and 17544 contact points. The rigid block used to simulate the shear type displacements imposed experimentally at the supports is shown in Figure 12a.

The obtained failure mechanism is shown in Fig. 12b. The result is in a good agreement with experimental outcomes, where a four hinges failure mechanism was observed, with opposite webs having inverted signs.

FIGURE 12. a) Cross vault dimensions; b) Cross vault collapse mechanism.

In Fig. 13 the collapse horizontal reactions at the moving support for the associative and non-associative solution are compared with the force/displacement curves presented in [62], the forces being expressed as a percentage the total weight *W*. The results show that the reaction at the moving support corresponding to the dilatant and non-dilatant behavior differ of about 20.5% and 7.1% from experimental values.

FIGURE 13. Cross vault: Comparison of numerical and experimental results [62].

5 APPLICATION TO A TWO-STOREY MASONRY BUILDING

The case study considered in this section is a two-storey masonry building comprising two front walls and two side walls with door and window openings at ground and first levels. The geometric configuration (Figs. 14a, 14b) was defined in order to be representative of an historic masonry building and to evaluate the ability of the developed software in predicting different failure mechanisms varying loading conditions, connection configurations and mechanical parameters. The length of the front and side walls are 8000 mm and 5000 mm. The wall thickness at ground and first levels are 500 mm and 300 mm, respectively. A barrel vault with 4000 mm span and 250 mm thick is modelled at the first level. The dimensions of the blocks used for discretization are 400 x 200 x 500 mm. The unit weight of masonry was taken as 18.0 kN/m^3 and the friction coefficient as 0.60. Floor loads at the second level were represented by vertical forces applied at each support block along the corresponding courses in the front walls and were calculated assuming a distributed load of 3.75 kN/m^2. The building is subject to horizontal loads expressed as a factor of dead loads and directed along the side walls.

In order to investigate the accuracy of the implemented software when different failure mechanisms are activated, two different configurations were considered for connections between the front and side walls: no connection (Fig. 14a) and interlocked connection (Fig. 14e).

On the basis of the previous geometric configurations, the following case studies were analyzed: *i*) no connection between front and side walls with tie-rods; *ii*) front and side walls interlocked; *iii*) front and side walls interlocked with tie-rods.

The tie-rods in configuration *i)* were modelled using additional rectangular blocks with dimensions 400 x 400 mm to represent the anchor plates, and applying a constant horizontal forces equal to 3.0 and 25.0 kN at each block.

FIGURE 14. Geometric model and collapse mechanisms configurations.

The predicted failure mechanisms and collapse load factors are shown in Figs. 14c-14f and Table 5. For the case studies *i*), the failure mechanisms involve simple overturning of the façade as expected, considering the connection configuration.

For the case *ii)* the overturning failure mechanism of the façade also involves portions of the sidewall due to interlocking effects.

In the case of the front walls interlocked with tie rods *iii)*, the predicted failure mode involves overturning of the façade with sidewalls.

The collapse load factors for the associative and non-associative behaviour corresponding to the predicted failure mechanism show that the influence of dilatancy is negligible, with the exception of the case *iv)*, where the difference between the load multiplier is about 9%. It was worth noting that the load factors computed for the configuration types *i)* closely match the analytical values obtained from the application of the virtual work principle to the macro-elements (i.e., the whole façade or the wall panel at the first level) .

A sensitivity analysis to friction coefficient was also carried out in the case of the building subject to horizontal loads along the longitudinal (y axis) direction to evaluate the effects on the failure mechanism and collapse load multipliers. Three failure mechanisms were observed from the numerical solution varying the friction coefficient. For values of the friction coefficient greater than 0.40, a simple overturning mechanism occurs. When the friction coefficient is comprised in the range 0.3-0.4 the façade is subjected to bending failure at the first level (Fig. 15). For lower values of μ, a pure sliding failure mechanism of the building at the first level is obtained. Figure 16 shows the difference of associative and non-associative solutions varying the friction coefficient. The results show that the formulation adopted is stable and robust even when unfeasible values of the friction coefficient are assumed.

Table 5. Collapse mechanisms for two-storey masonry building. Comparison of numerical and analytical results.

Model	Model size (b × c)	Associative solution		Non-associative solution	
		α_{ass}	CPU Time (sec)	$\alpha_{non\ ass}$	CPU Time (sec)
Simple overturning	2114 x 23548	0.008	2.3	0.008	11.9
Simple overturning with ties	2116 x 23580	0.067	2.5	0.067	10.4
Complex overturning	2130 x 24148	0.103	6.3	0.095	49.9
Complex overturning with ties	2132 x 24180	0.143	4.5	0.130	63.5

FIGURE 15. Failure mechanisms predicted for horizontal live loads along y-axis: a) μ=0.6; b) μ=0.3.

In Fig.17 the failure mechanism predicted imposing a vertical movement to end piers using an L shaped support block is shown. The reaction $\boldsymbol{f_s}$ calculated at the moving support for the non-associative solution is equal to 101.4 kN. Figure 17b also shows the distribution of cracks predicted on the barrel vault and the out-of-plane behaviour of the corner at the top of the building.

The solutions of the limit analysis problems took up to about 60 seconds, which is a quite short CPU time considering that in this case the numerical models generated comprise up to 2132 blocks and 24180 contact points.

FIGURE 16. a) Sensitivity analysis to friction coefficient μ; b) Collapse load factor vs. iteration for μ=0.6.

FIGURE 17. Predicted failure mechanism under support vertical movement.

6 CONCLUSIONS

A novel software tool with graphical user interface was implemented in Matlab for collapse mechanism analysis of three-dimensional masonry block assemblages subject to variable live loads and uniform settlement. The tool relies on a limit analysis formulation based on polyhedral rigid blocks interacting at no-tension, frictional interfaces. A point contact model is adopted to represent interactions at contact interfaces and cone programming is used for the solution of the associative and non-associative problems. The tool also interconnects with commercial software AutoCAD, so to

simplify the geometric generation of numerical models also in case of large scale rigid body assemblages with complex configuration and bond patterns.

The accuracy and computational efficiency of the developed tool were evaluated against a set of numerical and experimental case studies involving curved structures.

The comparison with numerical case studies from the literature showed that failure modes predicted with the developed tool are similar with those obtained using alternative contact formulations and mathematical programming solvers. In the interests of safety, it is worth noting that in few cases the obtained load factors are more than 15% lower than solutions from the literature.

The validation against experimental case studies also showed that predicted results are very similar to those obtained from testing outcomes, with percentage differences less than 17%.

The application to the numerical case study of a masonry building with a barrel vault confirmed that the developed tool can be used to model complex assemblages involving different structural elements.

Convergence characteristics of the iterative solution procedure implemented to take into account non-associative (i.e. non-dilatant) were stable and slightly influenced by mechanical and algorithm parameters and number of rigid blocks. Computational efficiency is also promising, with solution times up to about 60 seconds for the largest case study considered.

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