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On funicular gridshells and Airy stress functions

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Abstract

2D graphic statics is a method originating in the 19th century to analyse and design pin-jointed trusses using two reciprocal diagrams; the form diagram, which describes the structural geometry, and the force diagram, which describes the forces within the truss. This paper investigates '2.5D' graphic statics for gridshell design. This relies upon the form, force and slope diagrams. The force and slope diagrams combine, via a mixed area calculation, to give the funicular vertical load on a given node. In the same way that the force diagram can be constructed from a discrete Airy stress function, so can the slope diagram be constructed from a plane-faced gridshell. Both the plane-faced gridshell and Airy stress function are planar liftings of the form diagram. This new methodology allows the engineer to visually design the Airy stress function and gridshell simultaneously with the aim of producing gridshells which are both plane-faced and funicular with a given design load, such as a uniform projected loading. Whereas in 2D graphic statics one can control the form and force diagram, here one can control the Airy stress function and gridshell geometry. This paper also introduces the concept of mixed-Airy gridshells in which the Airy stress function and plane-faced gridshell are interchangeable for the same applied vertical load. Self-Airy gridshells, where the Airy stress function and gridshell are scaled versions of each other, are also introduced. Focus is given to designing plane-faced self-tied gridshells which do not thrust horizontally against the ground or surrounding structures.

Keywords: Maxwell, Airy, gridshell, funicular, graphic statics, self-stress

1. Introduction

2D graphic statics was pioneered in the 19th century by Maxwell, Rankine and Cremona [6]. It relies upon the reciprocal relationship between the *form* and *force* diagrams. The form diagram describes the geometry of the pin-jointed truss. The force diagram describes the forces within the truss. It is possible to construct the force diagram from the form diagram through a *discrete Airy stress function* [4]. This is a complete 3D plane-faced polyhedron which projects down onto the form diagram. This polyhedron, ϕ, has a dual polyhedron, ϕ^*, defined by a node in one mapping to a plane in the other and a plane in one mapping to a node in the other. The duality maps a point (α, β, γ) to the dual plane $z = \alpha x + \beta y - \gamma$. This is called the *pole and polar plane* and is a part of projective geometry which was well understood in the 19th century [8]. The projection of the dual polyhedron, ϕ^*, onto the horizontal plane is the force diagram. This is known as the Maxwell reciprocal diagram and a line in the form diagram has a corresponding perpendicular line in the force diagram whose length is the force in the bar. The concepts of 2D reciprocal diagrams can be taken to 3D with 4D Rankine stress functions and beyond into n-dimensions [6].

It has been argued [19] that the natural geometric setting for the Airy stress function is *isotropic geometry* which was pioneered by Strubecker [14]. Isotropic geometry possesses a *top-view* which is the projection of the 3D geometry onto the xy plane. This is closely related to the projective geometry which facilitated the initial advances in graphic statics. Isotropic geometry is particularly useful in the design of gridshells subjected to vertical loading – for non-vertical loading other methods, such as a full Rankine analysis [12], are required. In general 3D shell problems, loads normal to the shell surface are taken through curvature and loads in the tangent plane through changes in stress, but isotropic geometry simplifies this to a so-called 2.5D case for vertical loadings.

Plane-faced gridshells can also be described as polyhedrons. Taking the established 2D graphic statics methodology, it is possible to consider 2.5D shells and gridshells with the introduction of the *slope diagram*. Gridshells have already been studied using graphic statics; thrust network analysis uses 2D graphic statics in the design of masonry shells [3] and 3D Rankine reciprocals have been used to consider 3D space structures, including gridshells, but have struggled to find common usage due to the Rankine incompleteness problem [12]. The concepts introduced in this paper rely upon 2D graphic statics and differ from the thrust network analysis as the planarity of gridshell faces is built into the design process by construction.

Each linearly independent *lift* of the form diagram to a plane-faced polyhedron represents a state of self-stress of the 2D truss [4]. The number of mechanisms, m, and states of self-stress, s, in a 2D truss are related via the Maxwell-Calladine count [10]: $2v - b - r = m - s$. Here, v is the number of nodes, b is the number of bars and r is the number of restraints which is often taken as $r = 3$ (restraining rigid body motions). In this paper, s is a state of self-stress and S is a gridshell geometry. This paper investigates the funicular action of self-tied gridshells and the importance of states of self-stress for determining which vertical loads can be taken through funicular action (only axial forces, no bending moments).

This paper takes inspiration from the work of Vouga *et al* [19] who leveraged architectural geometry to consider self-supporting surfaces. The relationship between the isotropic curvature of the shell and Airy stress function for a funicular loading is given by equation (1) [19].

$$2K_{\phi} H_S^{rel} = F' \tag{1}$$

Here, K_{ϕ} is the isotropic Gaussian curvature of the Airy stress function ϕ, H_S^{rel} is the isotropic mean curvature of the gridshell, S, relative to ϕ. F' is the vertical loading per unit projected area on the gridshell. However, this equation can be difficult to use in design. Here, an engineering approach to the same problem is given with the intention of providing a clear design methodology for plane-faced funicular gridshells.

For the discussion which follows, it is important to define our terminology (Fig. 1 and 2).

- A *gridshell* is a collection of bars in 3-space which approximate a surface. The faces of the gridshell may or may not be plane.
- A *self-tied gridshell* is a gridshell which does not exert any horizontal loads on its supports under a given loading. This corresponds to the Airy stress function having its perimeter lying in one plane.
- A *plane-faced polyhedral gridshell* is a gridshell with plane faces throughout. It is a *lift* of the form diagram (projection of the gridshell onto the xy plane) [4]. This structure is labelled $\bar{S}$.
- A *funicular gridshell*, or a *self-supporting gridshell without bending*, is a gridshell which has only axial forces in its bars under its primary loading case (in this paper it is assumed that this is a uniform projected load). The gridshell, $\check{S}$, does not necessarily have plane faces. The axial forces in the gridshell relate to an Airy stress function, ϕ, which does have plane-faces. If a gridshell is both plane-faced and funicular, it is labelled S.

- *Self-Airy gridshells* are plane faced and funicular. They are defined by $S = \lambda\phi$, where λ is a scalar. *Uniform projected load self-Airy gridshells*, often abbreviated to *UPL self-Airy gridshells*, have the funicular load case as a uniform projected load so that the nodal load is proportional to the tributary area of the node in the form diagram. Tributary area has many interpretations, but in this paper the Voronoi area in the projection is used.
- *Mixed-Airy gridshells* are plane faced and funicular. The gridshell, S, has axial forces relating to a different Airy stress function, ϕ. This also indicates that the form diagram possesses multiple lifts.

This paper closely follows the nomenclature from Vouga *et al* [19] and uses $'$ to denote the projection onto the xy plane, or *top-view* in isotropic geometry [14]. The paper also uses * to denote the reciprocal. The authors endeavour to demonstrate the equivalency between isotropic geometry [14] and traditional engineering mechanics. Note that $S' = \phi'$ but $S^{*\prime} \neq \phi^{*\prime}$. The relationships between the above concepts are shown in Fig. 2.

The authors note that many other well-established form finding methods for shells and gridshells exist [1], including the force density method [15], dynamic relaxation and particle-spring systems. It is hoped that this new technique adds another arrow to the quiver of designers who might be interested in designing funicular plane-faced self-tied gridshells with functional architectural considerations built into the design loop.

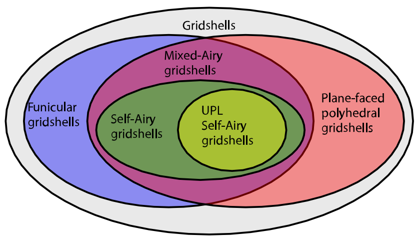


Figure 1: Venn diagram of definitions for gridshells under random vertical loads.

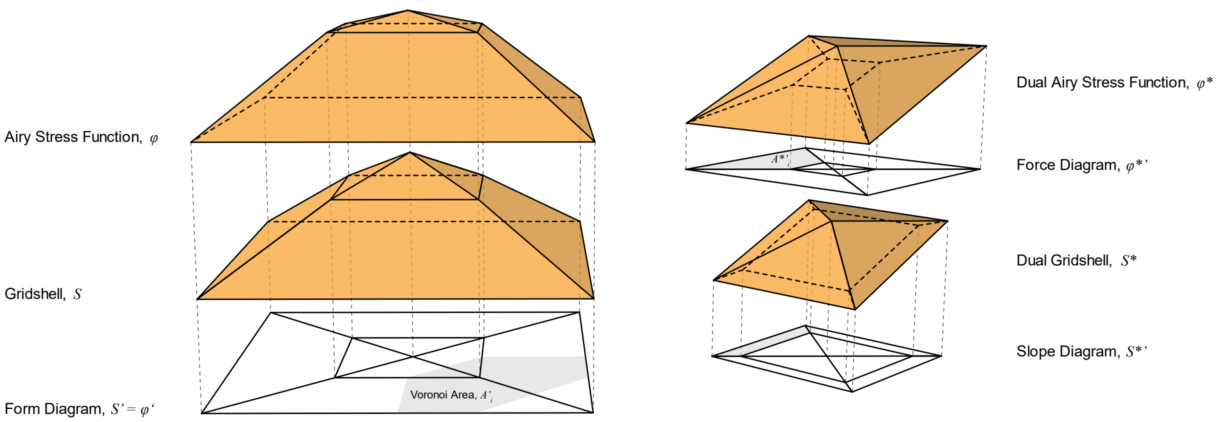


Figure 2: Plane faced gridshells and Airy stress functions are defined using these terms.
ϕ and ϕ^*, S and S^*, ϕ' and $\phi^{*\prime}$, S' and $S^{*\prime}$ are reciprocal pairs.

2. Mixed-Airy Gridshells

Consider a form diagram with multiple lifts so that $s > 1$. One lift of the form diagram is the plane-faced gridshell. There is an s-dimensional space of liftings which form plane-faced Airy stress functions which contain information on the balanced (self-tied) xy-projected axial forces in the gridshell. This means that there is an s-dimensional space of vertical external loads which the gridshell can equilibrate through pure funicular action. Plane faces are a desirable feature in gridshells for construction and cost reasons. Funicular action for dominant load cases is also desirable as it can reduce the flexural demand on the structure and the required structural quantities. This paper focuses on the dominant load case of a uniform projected load (quasi representing the self-weight of shallow shells or a uniform live or snow load).

2.1. Dual Airy Gridshells

Timoshenko [17] described the equilibrium of shells under gravity loads in equation (2). The shell, $z = S(x, y)$, and the applied vertical load, F', define the corresponding Airy stress function, $\phi = \phi(x, y)$; a membrane shell is statically determinate. An interesting feature of this equation is that if z and ϕ are swapped, the equation remains unchanged. Therefore, the shell and Airy stress function are interchangeable for the same vertical applied loading. This relationship is comparable to the form and force diagram in which either can be the structure and the other the forces.

$$\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = F' \tag{2}$$

This duality between shells and Airy stress functions extends beyond the continuum case and is also true in the discrete case [13]. This can be very powerful for finding the Airy stress function for a given roof geometry, and therefore the funicular gravity loads of the roof.

Take a plane-faced gridshell and label it ϕ_1. Consider it as the Airy stress function and use it to construct a force diagram. Use this to find the force density, $\omega = {}^{T}/_{L} = {}^{T'}/_{L'}$, in each bar. Note that this paper uses the Maxwell reciprocal diagram [7] so that a line in the force diagram is perpendicular to the corresponding line in the form diagram. The force density is preserved during a projection. From here, it is possible to use the force density method [15] to find a geometry, ϕ_2. The loads applied during the force density procedure are the vertical design loads. Due to the mixed-Airy knowledge, ϕ_2 is the roof if ϕ_1 is the Airy stress function and correspondingly ϕ_2 is the Airy stress function if ϕ_1 is the roof for the same applied vertical loading and the same vertical support forces. This assumes that ϕ_2 is plane-faced which is not necessarily true for general gridshell topologies. If the Airy stress function is not plane-faced, the lack of planarity indicates that the gridshell must contain bending moments to carry the applied loads [21]. A full investigation into how this non-planarity manifests is left for future research.

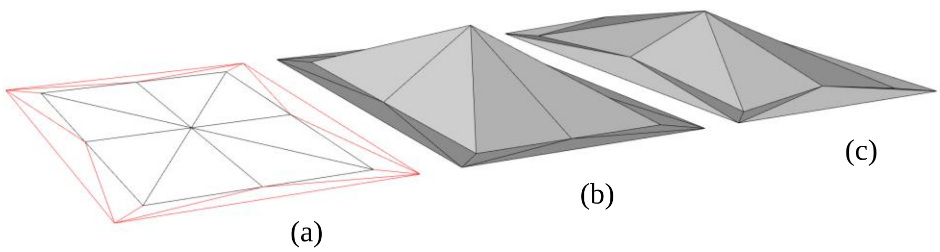


Figure 3: (a) A red restraining frame is added to the form diagram to take the horizontal forces (if any), from the roof structure members which are drawn in black.
(b) The roof with the restraining frame in the horizontal plane.
(c) The Airy stress function for roof (b) with equal vertical reactions at each perimeter support. Note that the restraining frame in (c) is not confined to the horizontal plane, indicating that it is stressed in the roof (b). Note that (b) and (c) are interchangeable; either one could be the structure and the other the Airy stress function for the same vertical loading. If (b) is the structure, it develops horizontal reactions against the restraining frame whereas (c) does not.

One can add a restraining frame to the perimeter of the gridshell in the form diagram to provide any external horizontal forces which can be exerted on the gridshell [10]. This restraining frame may not be built, but rather represents a system that can accommodate reactions, if any, on the surrounding structure.

The boundary conditions during the force density method are very important [15]. In this methodology, the designer has two options; they can either prescribe a vertical reaction force on an edge node in ϕ_2, or prescribe the vertical height of an edge node in ϕ_2. The vertical height of the node in ϕ_2 or the vertical reaction force at the node is then determined in each case respectively. For many design cases, the restraining frame is planar (lies in one plane) for the roof. However, it might not be planar in the Airy stress function which means that the exterior truss is stressed. If the designer chooses to set every perimeter node to the same height in the force density method, then the restraining truss is automatically planar and therefore unstressed as the Airy stress function has a planar perimeter. This means that the gridshell is self-tied.

2.2. Slope Diagram

The *slope diagram* is a novel concept which describes the slope, $\zeta = {}^{\Delta z}/_{L'}$, of a bar in a gridshell. The slope diagram allows the designer to intuitively see the isotropic Gaussian curvature of the shell ($K = \frac{A^{*\prime}}{A'}$ where $A^{*\prime}$ is the polygonal area in the slope diagram and A' is the tributary area of the node in the form diagram) and identify any areas which do not have enough curvature for buckling or other reasons. It also allows the concept of mixed areas to be introduced, as discussed later in this paper. It is noteworthy that $A^{*\prime}$ is an approximation of the angular deficit at a node.

Take a plane faced gridshell and consider it as an Airy stress function and construct a corresponding pseudo force diagram. This pseudo force diagram is the slope diagram of the gridshell. The slope of a bar in the gridshell is given by the perpendicular distance from the origin to the corresponding line in the slope diagram. This is shown by the red arrows in Fig. 4. The arrows from the origin to the line point in the uphill direction and provide the value of the slope of the bar, ζ. Consider the simple example, shown in Fig. 4. A node in the form diagram has valency of 4. The force and slope diagrams are both quads but are parallel redrawings of each other as they share the same form diagram [4].

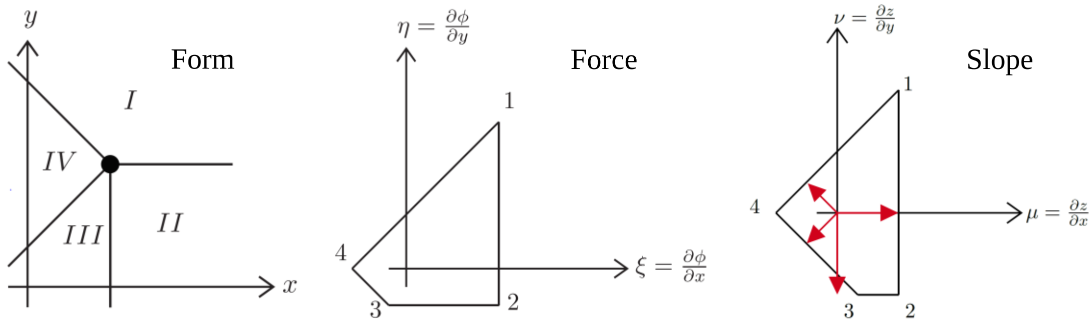


Figure 4: Simple example of the form, force and slope diagrams. The length of the line in the force diagram gives the horizontal force in the corresponding bar. The perpendicular distance of the line from the origin in the slope diagram gives the slope of the corresponding bar in the gridshell (length of red arrows).

Due to their construction, the force and slope diagrams are parallel redrawings of each other [4]. As the vertical force in a bar is $T_v = T'\zeta$, it can be found using the force and slope diagrams. The sum of these forces gives the negative of the vertical load which must be applied to a given node for equilibrium to be achieved.

$T_v = T'\zeta$ is closely related to the Minkowski sum of the force and slope diagram polygons. The Minkowski sum is a well-established concept in geometry, but in its simplest form involves tracing one polygon around the perimeter of the other, as shown in Fig. 5. When one does this with the force and slope diagram polygons, the area of a grey rectangle is equal to $T_v = T'\zeta$. The Minkowski sum of two polygons, B and C, is often written as $B + C$. The mixed area, $\hat{A}(B, C)$, is then given by $2\hat{A}(B, C) = A(B + C) - A(B) - A(C)$ where A is the area of the polygon. Twice the mixed area is the sum of all the grey areas in Fig. 5. As the equilibrium load on a node, F_i, is given by $F_i = \sum T_v$, then twice the mixed area of the force and slope polygons is equal to the equilibrium load on a node. This was described by Vouga *et al* [19]. It is possible to extend this method to consider warped funicular gridshells by triangulating warped faces with zero-force bars, as described in Millar *et al* [13].

$$2\hat{A}_i(z, \phi) = F' A_i' = F_i \tag{3}$$

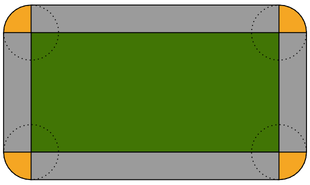


Figure 5: Minkowski sum of the green rectangle and the orange circle. Twice the mixed area is shown in grey and is independant of the order of mixing.

Twice the mixed area of the force and slope diagrams can be found by tracing the vectors from the origin to the line in the slope diagram (red arrows in Fig. 4) around the corresponding line in the force diagram. This produces a rectangle for each line, as shown in blue in Fig. 6. The area of each blue rectangle is the vertical force in the bar, $T_v = T'\zeta$. The area is signed (oriented area) to indicate whether the force exerted on a node by the bar is upwards or downwards. If the rectangle lies outside the force polygon then the force on the node is upwards. In the example shown in Fig. 6, all the rectangles lie outside the force polygon so exert an upwards force on the node. Therefore, the equilibrium load on the node is the sum of the rectangular areas and of the opposite sense. This diagram is called the *FS Minkowski Diagram* as it is closely related to the Minkowski sum of the force and slope diagram polygons.

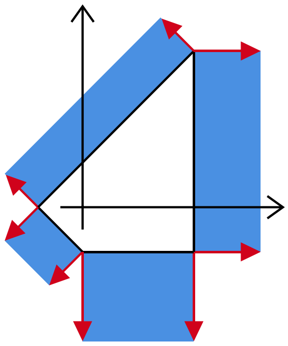


Figure 6: FS Minkowski Diagram. Twice the mixed area is shown in blue. The sum of oriented areas is equal to the vertical load on the gridshell node.

2.3. Funicular Load Space

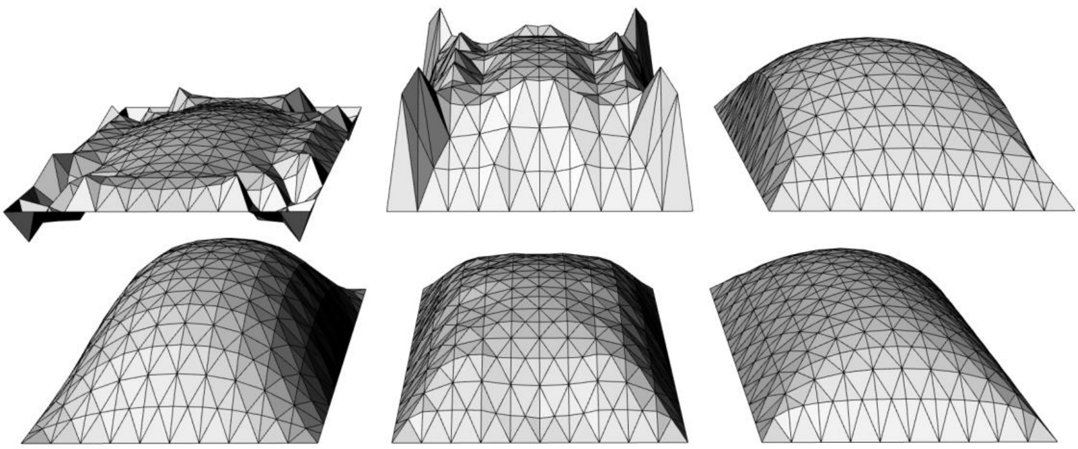
An interesting question in the light of mixed-Airy gridshells is how large is the space of funicular loads? Take a form diagram which possesses s lifts and, therefore, s states of self-stress. There are s linearly independent Airy stress functions and therefore s linearly independent mixed areas for a given roof geometry (which does not need to be plane faced). Therefore, the funicular load space has a size of s. If there are V internal nodes and all perimeter nodes are vertically supported, then the size of the load space which is not funicular, and therefore must be taken with bending, is $V - s$. If all load cases are funicular then $V = s$. This condition is met if the gridshell is fully triangulated and supported at the edge. However, fully triangulated gridshells could cause structural and construction issues at nodes which are seldom torsion-free (the members at a node do not share a common axis).

It is desirable to maximise the number of states of self-stress for a given connectivity. This is an area of ongoing research. It is also desirable that the funicular load cases cover the primary design load cases which often include self-weight (or a uniform projected load) and perhaps an antisymmetric load, such as a snow drift load. For a symmetric roof, it must possess a symmetric lift for it to support a symmetric load and an antisymmetric lift for it to support an antisymmetric load though funicular action only. However, even if the form possesses these lifts, it is not guaranteed that the specific design load is included in the funicular load space. *Designing* the funicular load space, and therefore the mixed-Airy gridshell, is discussed in the following Section 2.4.

2.4. Designing Triangulated Mixed-Airy Gridshells

The following describes some methods for the design of triangulated self-tied funicular gridshells using mixed-Airy methods – the triangulation of the form diagram avoids the need to discuss face planarity.

Fig. 7 shows a selection of roof geometries on the bottom with the corresponding Airy stress function for a uniform projected load above. The roof geometry, as well as the Airy stress function geometry, are very sensitive. The load path can be designed – the designer can choose how the applied load reaches the perimeter vertical supports. Different design principles can be included – smooth Airy stress function for smaller forces, or relatively uniform forces for easier node design, to mention just two. The Maxwell-Minkowski diagram [11] is very useful for visualising the sign and magnitude of the projected forces within the gridshell. From here, the designer has control of the forces and geometry. It is worth noting that Fig. 7c is based on the solutions of Williams [21] which are derived from an electro-magnetic analogy and the Biot-Savart law. This method resulted in a gridshell which has good engineering properties. This method allows the forces in a triangulated gridshell under any vertical load to be found without the need for finite element analysis.



(a) (b) (c)

Figure 7: Gridshells (bottom) and their Airy stress functions (top) for uniform projected loading. Note that because of the duality between the gridshell and its Airy stress function, the top row could be the gridshell and the bottom row would be the corresponding Airy stress functions.
(a) Negative Gaussian curvature at the corners induces tension into the corners.
(b) Teeth-like Airy-stress function along the perimeter indicates some truss action.
(c) A pillow shaped gridshell which contains a limited number of tension bars.

3. Self-Airy Gridshells

Some gridshells have a form diagram which possess only a single lift (state of self-stress) [4]. In this case, the Airy stress function and plane-faced gridshell, z, must be related by $z = \lambda\phi$, where λ is a scalar. This can also be true if the form diagram possesses multiple lifts. If this gridshell is in equilibrium with a uniform projected load then it is called a *UPL self-Airy gridshell*. This is a special case of the mixed-Airy loading where the slope and force diagrams are scaled versions of each other. The mixed area is now equal to the area of the polygon in the force diagram; $\hat{A}_i(z,\phi) = \lambda A_i^{*\prime}$ where $A_i^{*\prime}$ is the area of the polygon in the force diagram. The equation which governs the funicular action of the gridshell is (4). Here, A_i' is the tributary area of the node i in the form diagram – in this paper it is taken to be the Voronoi area associated with the node in the form diagram.

$$2\lambda \frac{A_i^{*\prime}}{A_i{}'} = F' \tag{4}$$

The discrete isotropic Gaussian curvature is defined by, $K = {A_i^{*\prime}}/{A_i{}'}$ [14], which makes equation (4) equivalent to equation (1) with $H_s^{rel} = \lambda$. Therefore, if F' is constant then a UPL self-Airy gridshell has constant isotropic Gaussian curvature; this has been studied extensively in the continuum case [16]. The same results can be obtained from Timoshenko's equation (2), as shown in (5).

$$K_\phi = \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x \partial y}\right)^2 = \frac{F'}{2\lambda} \tag{5}$$

For both constant and variable F', equation (5) is of the Monge-Ampère type which has been studied extensively [2]. Solutions can be found through a variety of methods, including intermediate integrals. The case when F' is variable can be useful when considering loads such as the self-weight of a steep-sided gridshell. The continuum solutions, when discretized correctly, can yield UPL self-Airy gridshells.

Returning to equation (4), for a uniform projected load, F', the Voronoi area of a node is proportional to the area of the force diagram polygon. One way this can be achieved is if the force diagram and Voronoi diagram of the form diagram are scaled version of each other. This is met if the gridshell is an *i-circular* mesh [14]. Such a gridshell has a form diagram in which every polygon is circumscribed by a circle, as shown in Fig. 8. The Voronoi diagram is constructed using perpendicular bisectors which all meet at the centre of this circle. As these lines are perpendicular to the lines in the form diagram, the Voronoi diagram is a possible force diagram. However, no information on the boundary conditions has been given and the Airy stress function relating to this force diagram may not be a complete polyhedron (its perimeter may not lie in one plane).

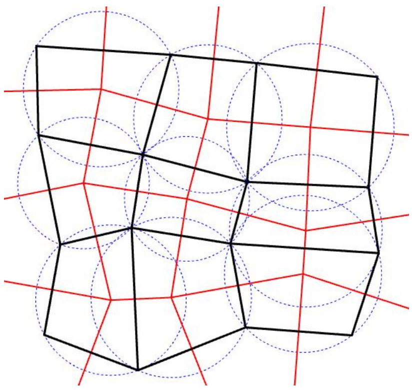


Figure 8: An i-circular mesh is a circular mesh in the form diagram. The mesh is shown in black with the dual/Voronoi diagram in red. The blue circles are the circumscribed circles of the form diagram.

3.1. Michell Dome

In his seminal paper on minimal material trusses, Michell described how bars in optimal trusses must meet at 90° in a truss-like continua [9]. Discrete Michell trusses are both circular and conical [9], so the authors took inspiration from these meshes to construct a gridshell. The geometry of a discrete Michell truss approximates a logarithmic spiral where the nodes of each polygon lie on circles [9]. Such a Michell dome is shown in Fig. 9 and is a UPL self-Airy gridshell. Finite element analysis of this gridshell under uniform projected loading shows essentially zero moment in all members, including the perimeter tension ring. The authors note that most engineering software calculates tributary areas using techniques other than the Voronoi diagram, so care is taken to ensure the correct design nodal loads are used.

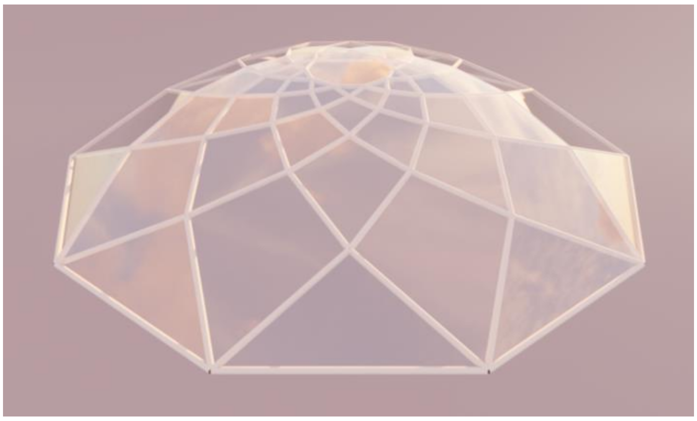


Figure 9: Michell Dome. This is a plane-faced quad-dominant gridshell which is funicular under uniform projected loading – it is a UPL self-Airy gridshell.

3.2. Schwarz-Christoffel Gridshell

The authors took the Schwarz-Christoffel mapping between a circle and a square [5] and used it as inspiration for a UPL self-Airy gridshell. The final gridshell is shown in Fig. 10. An initial grid based on the mapping was drawn and small alterations were made to make it circular. Multiple circular meshes were possible, but the authors chose one close to the original geometry. Enforcing a planar boundary, the gridshell possesses only a single lift. With this lift, the force diagram did not represent the Voronoi diagram of the form diagram, as shown in Fig. 11b. Therefore, a restraining frame was added to the perimeter of the form diagram. This introduced many additional lifts to the form diagram. With this, it was possible to lift the form diagram to produce a force diagram which was identical to the Voronoi diagram of the form diagram, as shown in Fig. 11a. As the initial geometry was close to being a UPL self-Airy, the lack of planarity of the perimeter in the final geometry was very small. For a dome $100m$ in diameter, the planarity error was only $38mm$. The structural behaviour is very sensitive to the Airy stress function geometry.

A finite element analysis of the gridshell was performed under uniform projected loading. There was essentially no bending moment in the members, although the tension ring had a small moment in it due to the non-planarity of this edge. However, the moment associated with this eccentricity was trivial. The authors note that as gridshells become more optimised, as in this case, the buckling load of the gridshell tends to decrease [1]. One possible explanation of this is because the bending stiffness of the gridshell decreases as it is not required to carry the dominant load cases. Further analysis by the designer should be performed to include all possible design loads, second-order analysis and global stability of the gridshell.

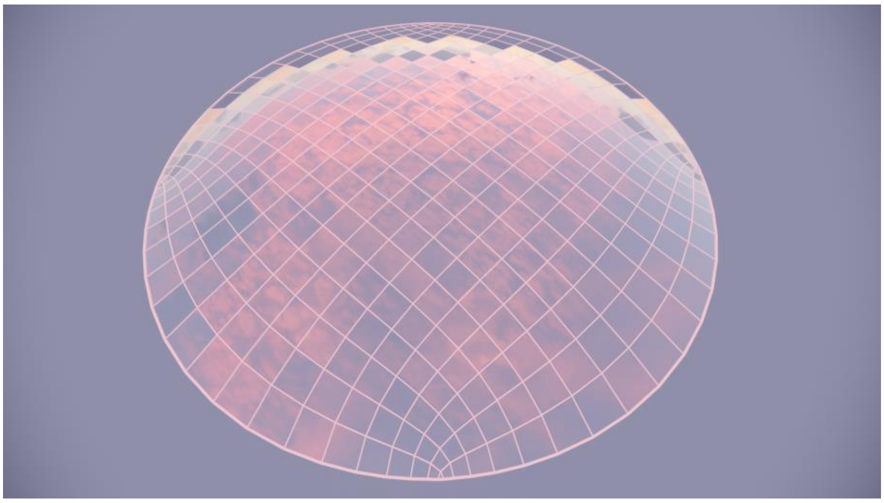
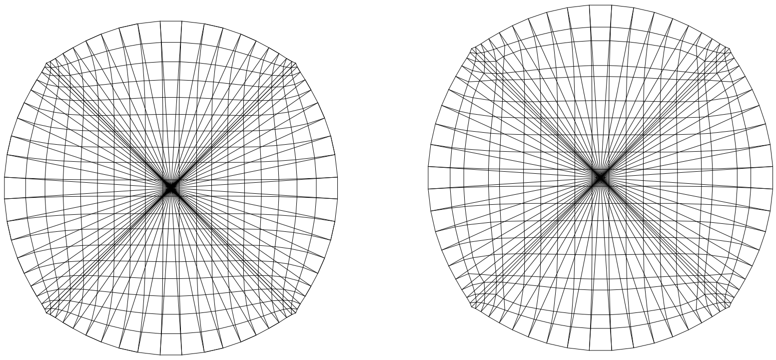


Figure 10: Schwarz-Christoffel Gridshell



(a) (b)

Figure 11: The force diagrams of the Schwarz-Christoffel gridshell. (a) UPL self-Airy. (b) Single lift. Note that near the perimeter of (b), the quads are not uniform in size and are not proportional to the Voronoi.

3.3. Strubecker Gridshell

Surfaces with constant isotropic Gaussian curvature are well studied [16]. As has been discussed, UPL self-Airy surfaces are defined by constant isotropic Gaussian curvature (see equation (5)). One example is a surface of revolution defined by $z = \int_0^r \sqrt{1 + r^2}$. Taking a section of this surface and applying a transformation which does not alter its isotropic curvature, one can obtain a surface patch which is a UPL shell. Once the umbilic points were identified, a suitable mesh was constructed – in this example, principal curvature lines were used to produce a quad-dominant discretization, however any planar discretization can be used. Further post-optimization using VaryLab [18] was performed to produce a plane-faced gridshell which was very close to being funicular under uniform projected loading. Considering this as the Airy stress function and using the force density method [15], one obtains a gridshell with slightly warped faces. Averaging these two meshes and then replanarizing in an iterative process gets the gridshell closer to funicular whilst maintain plane faces. This is an interesting example as the perimeter of the surface patch is not smooth.

A similar process of planar discretization followed by some post-optimization could be applied to many surfaces with constant isotropic Gaussian curvature. This allows many UPL self-Airy gridshells to be designed. The averaging technique can be used to create UPL self-Airy gridshells; when applied to the triangulated gridshell in Fig. 7c, one obtains a UPL self-Airy gridshell [13].

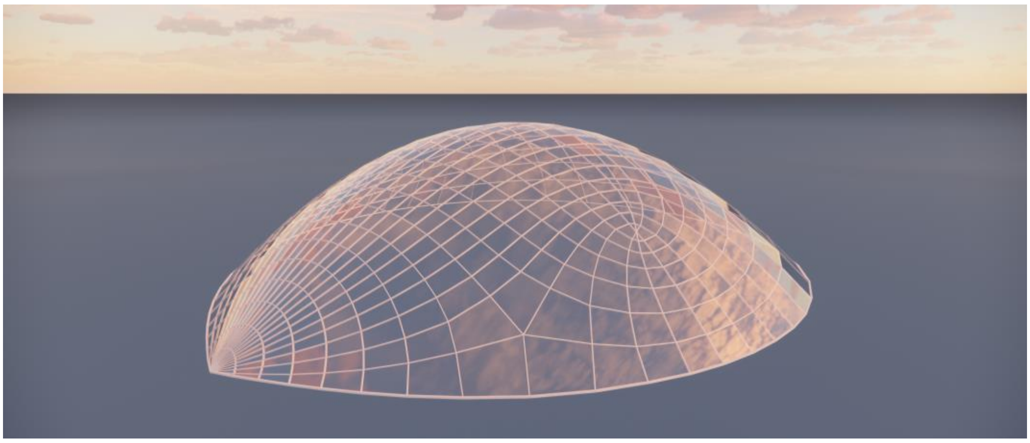


Figure 12: Strubecker gridshell

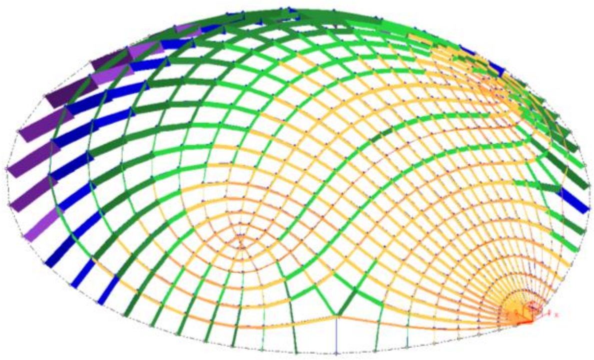


Figure 13: Axial force distribution in the Strubecker gridshell

The Strubecker gridshell has good engineering properties. Not only does it have plane quad faces for constructability, but the structural behaviour is also excellent. For a dome with a $50m$ span and $25m$ tall with interior $50mm$ square bars (edge members are $100mm$ square bars), under a $1kPa$ uniform projected load, the maximum deflection is $35mm$, the maximum force is $86.7kN$ (in the tension ring it is $300kN$), and the maximum moment is $1.38kNm$. The maximum eccentricity in any member is $0.288m$, which is comparably small.

4. Conclusions

This paper introduces some methods for the design of plane-faced funicular self-tied gridshells using the framework of graphic statics. The interplay between the geometry of gridshells and Airy stress functions has been explored. The novel concept of the slope diagram combines with the force diagram to allow for the design of funicular gridshells with plane faces. Inspired by isotropic geometry and architectural geometry, mixed-Airy and self-Airy gridshells have been introduced. The authors refer to a more mathematically complete paper [13] which develops some intricacies associated with this design approach.

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